Groups with only cyclic quotients

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Cyclic Groups

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$$G = \langle a \rangle = \{a^m \ : \ m \in \mathbb{Z}\}.$$

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Standard examples:

- $\mathbb{Z}$
- $\mathbb{Z}/m\mathbb{Z}$
- $\mu_n = \{z \in \mathbb{C} \ : \ z^n = 1\} \subset \mathbb{C}^*$

We will use C_n for a cyclic group of size n.

Basic Properties

Theorem

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Proof.

For (1), if $H \neq \{e\}$ then $H = \langle a^k \rangle$, where $k > 0$ is smallest such that $a^k \in H$.

For (2), $G/H = \{gH \ : \ g \in G\}$ and so

$$G/H = \left\{a^k H \ : \ k \in \mathbb{Z}\right\} = \left\{(aH)^k \ : \ k \in \mathbb{Z}\right\}.$$

Main question

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Do any other groups admit the same quotient result?

More precisely, are there any non-cyclic groups G such that the quotient G/N is cyclic for every non-trivial normal subgroup N?

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More precisely, are there any non-cyclic groups G such that the quotient G/N is cyclic for every non-trivial normal subgroup N?

Clearly the answer is yes, for the “vacuous” cases: if G is a simple group.

Main question rephrased

Definition

A group G is *just-non-cyclic* if G/N is cyclic for every non-trivial normal subgroup $N \leq G$.

Problem

Classify all just-non-cyclic groups.

For the most part, we will focus on finite groups.

Finite Abelian Groups

Non-examples

A non-cyclic, finite Abelian group $G \cong \prod_i C_{p_i^{e_i}}$ with $i \geq 3$ cannot be just-non-cyclic.

Proof.

For G to be non-cyclic, $p_i = p_j$ for some i and j. We may assume neither is p_1. Take $N = C_{p_1^{e_1}} \times \prod_{i>1} \{0\}$ so that $G/N \cong \prod_{i>1} C_{p_i^{e_i}}$. If $i > 3$ then the latter product still contains both components with $p_i = p_j$. □

Finite Abelian Groups continued

Proposition

If G is a p-group and just-non-cyclic then G is Abelian.

Proof.

$Z(G)$ is non-trivial, so G just-non-cyclic implies $G/Z(G)$ is cyclic and so G Abelian. □

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Proposition

A non-cyclic, finite Abelian group G is just-non-cyclic if and only if $G \cong C_p \times C_p$.

An infinite case

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Consider $G = \hat{\mathbb{Z}} := \varprojlim \mathbb{Z}/n\mathbb{Z}$.

If $N \leq \hat{\mathbb{Z}}$ has finite index, then $\hat{\mathbb{Z}}/N$ is cyclic.

- Can realize $\hat{\mathbb{Z}}$ as $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$.
- G/N would be a Galois group of $K/\mathbb{F}_p$, hence cyclic.

Symmetric Groups

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S_n is just-non-cyclic when $n \neq 4$.

Proof.

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Symmetric Groups

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- $n < 4$: trivial by group order considerations
- $n = 4$: $N \leq S_4$ isomorphic to the Klein 4-group, giving

$$S_4/N \cong S_3.$$

- $n > 4$: $A_n \leq S_n$ only non-trivial normal subgroups, and

$$S_n/A_n \cong C_2.$$

Dihedral groups

Proposition

The dihedral group $D_{2n} = \langle r, s \ : \ r^n = s^2 = e, \ srs = r^{-1} \rangle$ is just-non-cyclic if and only if $n = p$ a prime.

Proof.

- n prime: normal subgroup is $N = \langle r \rangle$, and

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Proof.

- n prime: normal subgroup is $N = \langle r \rangle$, and

$$G/N \cong C_2.$$

- n not prime: let $k \mid n$ and consider $N = \langle r^k \rangle \leq D_{2n}$. Then

$$D_{2n}/N \cong D_{2k}.$$

Semidirect Product

Definition

A semi-direct product $G \rtimes_{\varphi} H$ of two groups G and H, with respect to $\varphi : H \to \mathrm{Aut}(G)$, is the set $G \times H$ with

$$(g_1, h_1)(g_2, h_2) := (g_1 \varphi_{h_1}(g_2), h_1 h_2).$$

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$D_{2n} \cong C_n \rtimes_\varphi C_2$, with $\varphi_i(a) = a^i$.

Generalization

Proposition

$G = H \rtimes_{\varphi} C_n$ is just-non-cylic only if H is simple.

Proof.

If H is not simple, let $K \leq H$ be normal. Then $N = K \times \{0\}$ normal in G, and $G/N \cong (H/K) \rtimes_{\varphi} C_n$. □

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Fact

If p, q prime and $p \mid (q-1)$ then $C_q \rtimes_\varphi C_p$ is the unique non-Abelian group of order pq.

Question

Which of these are just-non-cyclic?

Next Steps?

Where an we go next? What ideas may be useful?

- Consider semi-direct products which involve other finite simple groups in the normal component.
- Distinguish between solvable groups and non-solvable groups? I have been told that "composition series" are useful to group theorists.
- Consider other kinds of products or other methods which relate to the general "extension" problem:

$$0 \to N \to G \to G/N \to 0.$$

Questions

Some related questions that I am curious about:

- What can one say when $|G|$ is infinite? Might representation theory be useful?
- Is a general procyclic group also just cyclic?
- Is there anything particularly interesting about asking this question of topological groups?
- Are just cyclic groups easily realizable as Galois groups? If so, how do fields $K/\mathbb{Q}$ with just cyclic Galois groups relate to cyclotomic fields and class field theory?